# AQA Level 2 Further mathematics Further algebra

## Section 4: Proof and sequences

### Solutions to Exercise

1. If $n$ is odd, $n^3$ and $n^2$ are both odd, so $n^3 - n^2$ is even.
   If $n$ is even, $n^3$ and $n^2$ are both even, so $n^3 - n^2$ is even.
   Therefore for all positive integers $n$, $n^3 - n^2$ is always even.

2. For three consecutive integers, at least one will be even, and one will be a multiple of 3. Therefore the product of the three integers is both a multiple of 2 and a multiple of 3, and so is a multiple of 6.

3. Any odd number can be written as $2n + 1$, where $n$ is an integer.
   So the square of an odd number can be written as $(2n+1)^2 = 4n^2 + 4n + 1$
   $= 4n(n+1)+1$
   One of $n$ and $n + 1$ is even, so $n(n+1)$ is a multiple of 2 and therefore $4n(n+1)$ is a multiple of 8.
   So $4n(n+1)+1$ is one more than a multiple of 8. So the square of any odd number is always 1 more than a multiple of 8.

4. (i) nth term $= 3n - 1$
   1st term $= 3\times 1 - 1 = 2$
   2nd term $= 3\times 2 - 1 = 5$
   3rd term $= 3\times 3 - 1 = 8$
   4th term $= 3\times 4 - 1 = 11$
   Sequence is 2, 5, 8, 11, ....

   (ii) nth term $= n^2 - 1$
   1st term $= 1^2 - 1 = 0$
   2nd term $= 2^2 - 1 = 4 - 1 = 3$
   3rd term $= 3^2 - 1 = 9 - 1 = 8$
   4th term $= 4^2 - 1 = 16 - 1 = 15$
   Sequence is 0, 3, 8, 18, ....

   (iii) nth term $= 3n^2 - 2n + 1$
   1st term $= 3\times 1^2 - 2\times 1 + 1 = 3 - 2 + 1 = 2$
   2nd term $= 3\times 2^2 - 2\times 2 + 1 = 12 - 4 + 1 = 9$
   3rd term $= 3\times 3^2 - 2\times 3 + 1 = 27 - 6 + 1 = 22$
   4th term $= 3\times 4^2 - 2\times 4 + 1 = 48 - 8 + 1 = 41$
   Sequence is 2, 9, 22, 41, ....

# AQA FM Further algebra 4 Exercise solutions

5. (i) Each term increases by 3, so the general term must involve 3n.
   nth term $= 3n - 1$.

   (ii) Each term decreases by 2, so the general term must involve -2n.
   nth term $= 12 - 2n$

6. (i) The sequence has nth term $an^2 + bn + c$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Terms | 3 | | 9 | | 17 | | 27 | | 39 |
| Differences | | 6 | | 8 | | 10 | | 12 | |
| Second differences | | | 2 | | 2 | | 2 | | |

So $a = 1$

| | | | | | |
|---|---|---|---|---|---|
| Terms | 3 | 9 | 17 | 27 | 39 |
| $an^2$ | 1 | 4 | 9 | 16 | 25 |
| $bn + c$ | 2 | 5 | 8 | 11 | 14 |

The values of $bn + c$ go up by 3 each time, so $b = 3$, and $c = -1$

The nth term of the sequence is $n^2 + 3n - 1$.

(ii) The sequence has nth term $an^2 + bn + c$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Terms | -2 | | 4 | | 14 | | 28 | | 46 |
| Differences | | 6 | | 10 | | 14 | | 18 | |
| Second differences | | | 4 | | 4 | | 4 | | |

So $a = 2$

| | | | | | |
|---|---|---|---|---|---|
| Terms | -2 | 4 | 14 | 28 | 46 |
| $an^2$ | 2 | 8 | 18 | 32 | 50 |
| $bn + c$ | -4 | -4 | -4 | -4 | -4 |

The values of $bn + c$ are all the same, so $b = 0$, and $c = -4$

The nth term of the sequence is $2n^2 - 4$.

(iii) The sequence has nth term $an^2 + bn + c$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Terms | 7 | | 12 | | 15 | | 16 | | 15 |
| Differences | | 5 | | 3 | | 1 | | -1 | |
| Second differences | | | -2 | | -2 | | -2 | | |

So $a = -1$

| | | | | | |
|---|---|---|---|---|---|
| Terms | 7 | 12 | 15 | 16 | 15 |
| $an^2$ | -1 | -4 | -9 | -16 | -25 |
| $bn + c$ | 8 | 16 | 24 | 32 | 40 |

The values of $bn + c$ go up by 8 each time, so $b = 8$, and $c = 0$

The nth term of the sequence is $-n^2 + 8n$.

7. (i) nth term $= \dfrac{2n+5}{4n-1}$

$1^{st}$ term $= \dfrac{2\times 1+5}{4\times 1-1} = \dfrac{2+5}{4-1} = \dfrac{7}{3}$

$5^{th}$ term $= \dfrac{2\times 5+5}{4\times 5-1} = \dfrac{10+5}{20-1} = \dfrac{15}{19}$

$100^{th}$ term $= \dfrac{2\times 100+5}{4\times 100-1} = \dfrac{200+5}{400-1} = \dfrac{205}{399}$

As $n \to \infty$, $2n+5 \to 2n$, $\quad 4n-1 \to 4n$

$$\text{so } \frac{2n+5}{4n-1} \to \frac{2n}{4n} = \frac{1}{2}$$

The limit of the sequence is $\frac{1}{2}$.

(ii) nth term $= \dfrac{1-6n}{2n+3}$

$1^{st}$ term $= \dfrac{1-6\times 1}{2\times 1+3} = \dfrac{1-6}{2+3} = -\dfrac{5}{5} = -1$

$5^{th}$ term $= \dfrac{1-6\times 5}{2\times 5+3} = \dfrac{1-30}{10+3} = -\dfrac{29}{13}$

$100^{th}$ term $= \dfrac{1-6\times 100}{2\times 100+3} = \dfrac{1-600}{200+3} = -\dfrac{599}{203}$

As $n \to \infty$, $1-6n \to -6n$, $\quad 2n+3 \to 2n$

$$\text{so } \frac{1-6n}{2n+3} \to \frac{-6n}{2n} = -3$$

8. <u>Method 1</u>

$n^2 + 2n - 5 = 1000$

$n^2 + 2n = 1005$

$n^2 + 2n + 1 = 1006$

$(n+1)^2 = 1006$

$n+1$ is a whole number but 1006 is not a square number so 1000 cannot be a term in the sequence.

<u>Method 2</u>

$n^2 + 2n - 5 = 1000$

$n^2 + 2n = 1005$

$n(n+2) = 1005$

$n$ and $n+2$ are consecutive even or consecutive odd numbers. To multiply to make 1005, they must both be odd.
1005 = 3 x 5 x 67
There are no consecutive odd numbers that multiply to make 1005 so 1000 cannot be a term in the sequence.

Method 3

$$n^2 + 2n - 5 = 1000$$

$n^2 + 2n - 1005 = 0$
Solve the quadratic equation.
$n = 30.72$ or $-32.72$
$n$ is not an integer so 1000 is not a term in the sequence.

9. (a) One possible sequence is $\dfrac{3n}{n+1}$

(b) One possible sequence is $4 - \dfrac{n}{n+1}$