AQA Level 2 Further mathematics Further algebra

Section 3: Inequalities and indices

Notes and Examples

These notes contain subsections on

Inequalities

Inequalities are similar to equations, but instead of an equals sign, =, they involve one of these signs:

$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

This means that whereas the solution of an equation is a specific value, or two or more specific values, the solution of an inequality is a range of values.

Inequalities can be solved in a similar way to equations, but you do have to be very careful, as in some situations you need to reverse the inequality. This is shown in these examples.

Linear inequalities

A linear inequality involves only terms in x and constant terms.

Example 1

Solve the inequality $3x+1>x-5$

You can treat this just like a linear equation.

Solution

$3x+1>x-5$

Subtract x from each side

$2x+1>-5$

Subtract 1 from each side

$2x>-6$

Divide both sides by 2

$x>-3$

The next example involves a situation where you have to divide by a negative number. When you are solving an equation, multiplying or dividing by a negative number is not a problem. However, things are different with inequalities.

The statement $-3 < 2$ is clearly true.

If you add something to each side, it is still true

$-3 + 1 < 2 + 1$
$-2 < 3$ ✓

If you subtract something from each side, it is still true

$-3 - 4 < 2 - 4$
$-7 < -2$ ✓

If you multiply or divide each side by a positive number, it is still true

$-3 \times 2 < 2 \times 2$
$-6 < 4$ ✓

However, if you multiply each side by a negative number then things go wrong!

$-3 \times -2 < 2 \times -2$
$6 < -4$ ✗

When you multiply or divide each side by a negative number, you must reverse the inequality.

The following example demonstrates this. Two solutions are given: in the first the inequality is reversed when dividing by a negative number, in the second this situation is avoided by a different approach.

Example 2

Solve the inequality $1 - x \geq 2x - 5$

Solution (1)

$1 - x \geq 2x - 5$

Subtract $2x$ from each side

$1 - 3x \geq -5$

Subtract 1 from each side

$-3x \geq -6$

Divide both sides by -3, reversing the inequality.

$x \leq 2$

Solution (2)

$1 - x \geq 2x - 5$

Add x to each side

$1 \geq 3x - 5$

Add 5 to each side

$6 \geq 3x$

Divide both sides by 3

$2 \geq x$

Finish by writing the inequality the other way round.

$x \leq 2$

You can check that you have the sign the right way round by picking a number within the range of the solution, and checking that it satisfies the original inequality. In the above example, you could try $x = 1$. In the original inequality you get $0 \geq -3$, which is correct.

The ***Inequalities Activity*** takes you through the ideas behind some of the different methods of solving linear inequalities, including thinking about graphs.

You can look at some more examples using the Flash resource ***Linear inequalities***.

For more practice in solving linear inequalities, try the interactive questions ***Solving linear inequalities***.

There is also an ***Inequalities puzzle***, in which you need to cut out all the pieces and match linear inequalities with their solutions to form a large hexagon.

Quadratic inequalities

You can solve a quadratic inequality by factorising the quadratic expression, just as you do to solve a quadratic equation. This tells you the boundaries of the solutions. The easiest way to find the solution is then to sketch a graph.

Example 3

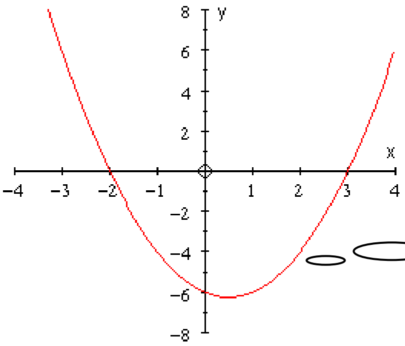
Solve the inequality $x^2 - x - 6 < 0$

Solution

$$x^2 - x - 6 < 0$$

$$(x+2)(x-3) < 0$$

This shows that the graph of $y = x^2 - x - 6$ cuts the x-axis at $x = -2$ and $x = 3$. Use this information to sketch the graph.



The solution to the inequality is the negative part of the graph. This is the part between –2 and 3.

The solution is $\quad -2 < x < 3$

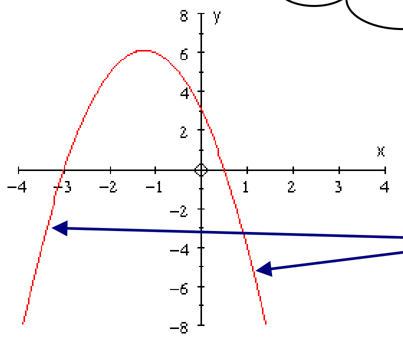
Example 4

Solve the inequality $3 - 5x - 2x^2 \leq 0$

Solution

$$3-5x-2x^2 \le 0$$
$$(3+x)(1-2x) \le 0$$

This shows that the graph of $y = 3-5x-2x^2$ cuts the x-axis at x = -3 and x = ½. You can now sketch the graph – note that as the term in x^2 is negative, the graph is inverted.



The solution to the inequality is the negative part of the graph. This is in fact two separate parts.

The solution is $x \le -3$ or $x \ge ½$.

Note: if you prefer to work with a positive x^2 term, you can change all the signs in the original inequality and reverse the inequality, giving $2x^2+5x-3 \ge 0$. The graph will then be the other way up, and you will take the positive part of the graph, so the solution will be the same.

To see more examples, use the Flash resource ***Quadratic inequalities***. (This shows an alternative approach using a number line.)

You can also look at the ***Solving inequalities video***, which uses a range of approaches.

For more practice in solving quadratic inequalities, try the interactive questions ***Solving quadratic inequalities***.

There is also a ***Quadratic inequalities puzzle***, in which you need to cut out all the pieces and match linear inequalities with their solutions to form a large triangle.

Multiplying expressions

The example below illustrates multiplying expressions involving indices.

Example 5

Simplify the expression $2xy \times 3yz^2 \times 4x^2z$.

Solution

$$2xy \times 3yz^2 \times 4x^2z = 2\times3\times4\times x\times x\times x^2\times y\times y\times z^2\times z$$
$$= 24x^3y^2z^3$$

You may be happy to do this in your head, without writing out the intermediate line of working.

For practice in examples like this one, try the interactive resource ***Simplifying products***.

When you are multiplying expressions like the ones in Example 5, you are using one of the rules of indices.

The rules of indices

Three rules of indices are:

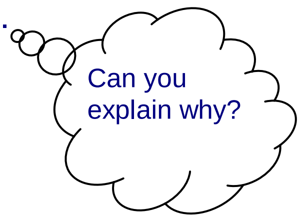
$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$\left(a^m\right)^n = a^{mn}$$

You can investigate these rules and see why they work by trying them out with simple cases, writing the sums out in full:
E.g., to demonstrate rule 3:

$$\left(2^3\right)^2 = \left(2\times2\times2\right)^2$$
$$= \left(2\times2\times2\right)\times\left(2\times2\times2\right)$$
$$= 2\times2\times2\times2\times2\times2$$
$$= 2^6$$
$$= 2^{3\times2}$$

Try some for yourself.

The number being raised to a power (a in this case) is called the **base**.
Note:
You can only apply these rules to numbers involving the same base.
So, for example, you cannot apply the rules of indices to $2^3 \times 3^5$.



Example 6
Simplify

(i) $a^4 \times a^7$ (ii) $x^9 \div x^4$ (iii) $(y^3)^6$ (iv) $2^3 \times 4^3$

Solution

(i) $a^4 \times a^7 = a^{4+7}$ using the first rule

$= a^{11}$

(ii) $x^9 \div x^4 = x^{9-4}$ using the second rule

$= x^5$

(iii) $(y^3)^6 = y^{3\times 6}$ using the third rule

$= y^{18}$

(iv) $2^3 \times 4^3 = 2^3 \times (2^2)^3$ At first sight this looks as if it cannot be simplified, as the bases are different. However, 4 can be written as a power of 2.

$= 2^3 \times 2^6$

$= 2^{3+6}$

$= 2^9$

You can see some similar examples using the Flash resource ***Laws of indices***.

Negative indices

There are two more rules, which follow from the three already introduced:

$$a^{-n} = \frac{1}{a^n}$$

$$a^0 = 1$$

Again, it's worth experimenting with numbers to get a feel for how and why these rules work.
e.g.

$$2^{-1} = \tfrac{1}{2}$$

$$\Rightarrow 2^1 \times 2^{-1} = 2 \times \tfrac{1}{2} = 1$$

And from rule 3,

$$2^1 \times 2^{-1} = 2^{1-1} = 2^0 = 1$$

Try some for yourself.

Note that it m … his were not so, the other rules would be inconsistent. If you consider a graph of $y = a^x$, for different values of a, you will see that it is perfectly natural that $a^0 = 1$. Try this on your graphical calculator.

Example 7
Find, as fractions or whole numbers,
(i) 2^{-4} (ii) 5^{-2} (iii) 3^0

Solution
(i) $2^{-4} = \frac{1}{2^4} = \frac{1}{16}$
(ii) $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$
(iii) $3^0 = 1$

You can see more examples like the ones above using the Flash resource ***Zero, negative and fractional indices***, choosing just the first two rules for now.

Fractional indices

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

Although these are equivalent, it is usually easier to use the first form, working out the root first so that you are dealing with smaller numbers.

As before, try experimenting with numbers to get a feel for how an why these rules work.

Example 8
Find, as fractions or whole numbers,
(i) $8^{\frac{1}{3}}$ (ii) $9^{\frac{3}{2}}$ (iii) $25^{-\frac{1}{2}}$ (iv) $4^{-\frac{5}{2}}$

Solution
(i) $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$
(ii) $9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27$
(iii) $25^{-\frac{1}{2}} = \frac{1}{25^{\frac{1}{2}}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$
(iv) $4^{-\frac{5}{2}} = \frac{1}{4^{\frac{5}{2}}} = \frac{1}{(4^{\frac{1}{2}})^5} = \frac{1}{(\sqrt{4})^5} = \frac{1}{2^5} = \frac{1}{32}$

You can see more examples like the ones above using the Flash resource **Zero, negative and fractional indices**, choosing the last of the rules. You can also look at the ***Indices video***.

You might be asked to solve equations involving indices. Just as when you solve a linear equation, you need to think about using inverse operations. So if the equation involves x^2, you need to take the square root. The same idea applies to more complicated indices.

Example 9

Solve the equations

(i) $x^{-\frac{2}{3}} = 4$ (ii) $x^5 = 4\sqrt{2}$

Solution

(i) $x^{-\frac{2}{3}} = 4$

$$\left(x^{-\frac{2}{3}}\right)^{-\frac{3}{2}} = 4^{-\frac{3}{2}}$$

The inverse of raising x to the power $-\frac{2}{3}$ is to raise to the power $-\frac{3}{2}$

$$x = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{2^3} = \frac{1}{8}$$

(ii) $x^5 = 4\sqrt{2}$

$$x^5 = 2^2 \times 2^{\frac{1}{2}} = 2^{\frac{5}{2}}$$

$$(x^5)^{\frac{1}{5}} = (2^{\frac{5}{2}})^{\frac{1}{5}}$$

$$x = 2^{\frac{1}{2}} = \sqrt{2}$$

For further practice in manipulating indices, there are three puzzles in which you need to match equivalent expressions to form a large hexagon. There is a ***numeric indices puzzle***, an ***advanced numeric indices puzzle*** (more difficult examples) and an ***algebraic indices puzzle*** (in which the expressions to be manipulated are algebraic).