AQA Level 2 Further mathematics Further algebra

Section 2: Further equations

Notes and Examples

These notes contain subsections on

- Linear simultaneous equations using elimination
- Linear simultaneous equations using substitution
- One linear and one quadratic equation
- The factor theorem
- Dividing a cubic expression by a linear factor

Linear simultaneous equations using elimination

This work may be revision. You need to make sure that you can solve linear simultaneous equations confidently before you move on to the work on one linear and one quadratic equation, which will probably be new to you.

Simultaneous equations involve more than one equation and more than one unknown. To solve them you need the same number of equations as there are unknowns.

One method of solving simultaneous equations involves adding or subtracting multiples of the two equations so that one unknown disappears. This method is called *elimination*, and is shown in the next example.

Example 1

Solve the simultaneous equations

$$3p + q = 5$$
$$p - 2q = 4$$

Solution

① $3p + q = 5$

② $p - 2q = 4$

Adding or subtracting these equations will not eliminate either p or q. However, you can multiply the first equation by 2, and then add. This will eliminate p.

① × 2 $6p + 2q = 10$

② $p - 2q = 4$

Adding: $7p = 14$

$p = 2$

$3 \times 2 + q = 5$

$6 + q = 5$

$q = -1$

Now substitute this value for p into one of the original equations – you can use either, but in this example, ① is used.

The solution is $p = 2$, $q = -1$.

Notice that, in Example 1, you could have multiplied equation ② by 3 and then subtracted. This would give the same answer.

Sometimes you need to multiply each equation by a different number before you can add or subtract. This is the case in the next example.

Example 2
5 pencils and 2 rubbers cost £1.50
8 pencils and 3 rubbers cost £2.35
Find the cost of a pencil and the cost of a rubber.

Solution

① $5p+2r=150$

② $8p+3r=235$

> Let p represent the cost of a pencil and r represent the cost of a rubber. It is easier to work in pence.

① × 3 $15p+6r=450$

② × 2 $16p+6r=470$

> The easiest method is to multiply equation ① by 3 and equation ② by 2. (You could of course multiply ① by 8 and ② by 5).

Subtracting: $-p=-20$

$p=20$

$5\times 20+2r=150$

> Substitute this value of p into equation ①

$100+2r=150$

$2r=50$

$r=25$

A pencil costs 20p and a rubber costs 25p.

Linear simultaneous equations using substitution

An alternative method of solving simultaneous equations is called substitution. This can be the easier method to use in cases where one equation gives one of the variables in terms of the other. This is shown in the next example.

Example 3
Solve the simultaneous equations

$3x-2y=11$

$y=5-2x$

Solution

> Substitute the expression for y given in the second equation, into the first equation.

$3x - 2(5 - 2x) = 11$

$3x - 10 + 4x = 11$

$7x = 21$

$x = 3$

Multiply out the brackets

$y = 5 - 2 \times 3$

$= 5 - 6$

$= -1$

Substitute the value for x into the original second equation

The solution is $x = 3$, $y = -1$

For some practice in examples like the ones above, try the interactive resources ***Solving linear simultaneous equations*** and ***Forming and solving linear simultaneous equations***.

You can also look at the ***Simultaneous equations video***.

One linear and one quadratic equation

When you need to solve a pair of simultaneous equations, one of which is linear and one of which is quadratic, you need to substitute the linear equation into the quadratic equation.

Example 4

Solve the simultaneous equations

$x^2 + 2y^2 = 6$

$x - y = 1$

Solution

Start by using the linear equation to write one variable in terms of the other.

$x = y + 1$

Now substitute this expression for y into the first equation

$(y+1)^2 + 2y^2 = 6$

$y^2 + 2y + 1 + 2y^2 = 6$

Multiply out, simplify and factorise

$3y^2 + 2y - 5 = 0$

$(3y+5)(y-1) = 0$

$y = -\frac{5}{3}$ or $y = 1$

Sometimes you will need to use the quadratic formula to solve the resulting quadratic equation.

$y = -\frac{5}{3}$ $\quad x = y + 1 = -\frac{5}{3} + 1 = -\frac{2}{3}$

$y = 1$ $\quad x = y + 1 = 1 + 1 = 2$

Now substitute each value for y into the linear equation to find the corresponding values of x

The solutions are $x = -\frac{2}{3}, y = -\frac{5}{3}$ and $x = 2, y = 1$

You can look at some more examples like these using ***Simultaneous equations***.

The factor theorem

You already know that you can solve some quadratics by factorising them.

e.g.	to solve the quadratic equation	$x^2 + 3x - 10 = 0$
	you factorise:	$(x + 5)(x - 2) = 0$
	and deduce the solutions	$x = -5$ and $x = 2$

Clearly, for $f(x) = x^2 + 3x - 10$, $f(-5) = 0$ and $f(2) = 0$.
$(x + 5)$ is a factor of $f(x) \Leftrightarrow f(-5) = 0$
$(x - 2)$ is a factor of $f(x) \Leftrightarrow f(2) = 0$

This idea can be extended to other polynomials such as cubics.

For example, for the cubic function $g(x) = (x-1)(x+2)(x-3)$, $g(1) = 0$, $g(-2) = 0$ and $g(3) = 0$.
$(x - 1)$ is a factor of $g(x) \Leftrightarrow g(1) = 0$
$(x + 2)$ is a factor of $g(x) \Leftrightarrow g(-2) = 0$
$(x - 3)$ is a factor of $g(x) \Leftrightarrow g(3) = 0$

In general, the **factor theorem** states that:

> If $(x-a)$ is a factor of $f(x)$, then $f(a) = 0$ and $x = a$ is a root of the equation $f(x) = 0$.
> Conversely, if $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$.

The factor theorem is useful for factorising cubic expressions and for solving cubic equations.

As with quadratics, it may or may not be possible to factorise a cubic expression. If it is possible, the first step is to factorise it into a linear factor and a quadratic factor. Then it may be possible to factorise the quadratic factor into two further linear factors.

Example 5
(i) Show that $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$.
(ii) Show that $x^3 - 2x^2 - 5x + 6 = (x + 2)(x^2 - 4x + 3)$.
(iii) Hence solve the equation $x^3 - 2x^2 - 5x + 6 = 0$.

Solution
(i) $f(x) = x^3 - 2x^2 - 5x + 6$

$$f(-2) = (-2)^3 - 2(-2)^2 - 5\times -2 + 6$$
$$= -8 - 8 + 10 + 6$$
$$= 0$$

f(-2) = 0 so (x + 2) is a factor of f(x).

(ii) $(x+2)(x^2-4x+3) = x(x^2-4x+3) + 2(x^2-4x+3)$
$$= x^3 - 4x^2 + 3x + 2x^2 - 8x + 6$$
$$= x^3 - 2x^2 - 5x + 6$$

(iii) $x^3 - 2x^2 - 5x + 6 = 0$
$$(x+2)(x^2-4x+3) = 0$$
$$(x+2)(x-1)(x-3) = 0$$

The quadratic factor can be factorised into two linear factors

The roots of the equation are x = -2, x = 1 and x = 3.

In the example above, you were given the factorisation into a linear and a quadratic factor, and you just needed to show that it was correct. More often you will need to find the quadratic factor yourself. So you need a method of dividing a cubic expression by a linear factor to give the quadratic factor.

There are several different methods of doing this.

Dividing a cubic expression by a linear factor

In this course, all the cubic expressions you will see in this course have 1 as the coefficient of x^3. This means that the quadratic factor will have 1 as the coefficient of x^2. So you can express the quadratic factor as $x^2 + px + q$, and then multiply out and find the values of p and q by comparing with the original expression.

Example 6

(i) Show that $x - 1$ is a factor of $x^3 - 3x^2 - x + 3$.

(ii) Factorise $x^3 - 3x^2 - x + 3$ completely.

Solution

(i) f(x) = $x^3 - 3x^2 - x + 3$

f(1) = 1 – 3 – 1 + 3 = 0

so $x - 1$ is a factor.

(ii) $x^3 - 3x^2 - x + 3 = (x-1)(x^2 + px + q)$
$$= x(x^2 + px + q) - 1(x^2 + px + q)$$
$$= x^3 + px^2 + qx - x^2 - px - q$$
$$= x^3 + (p-1)x^2 + (q-p)x - q$$

Equating coefficients of x^2 $\Rightarrow p - 1 = -3 \Rightarrow p = -2$
Equating constant terms $\Rightarrow -q = 3 \Rightarrow \Rightarrow q = -3$

(Check: Coefficients of $x = q - p = -3 - (-2) = -1$)

$$\begin{aligned} x^3 - 3x^2 - x + 3 &= (x-1)(x^2 - 2x - 3) \\ &= (x-1)(x+1)(x-3) \end{aligned}$$

In the example above, you may have noticed that it is very easy to find the value of q, just by thinking about the constant term. It is possible to do the factorisation 'in your head', without writing it out using p and q.

It's also clear that the constant term in the quadratic factor must be -3, to give the constant term 3 using -1 × -3 = 1.

So the factorisation can be written like this:

$$x^3 - 3x^2 - x + 3 = (x-1)(x^2 + \square x - 3)$$

You can then work out the missing coefficient by thinking about either the terms in x^2 or the terms in x. Multiplying out the linear and quadratic factors will give a term of $-x^2$ (from multiplying -1 in the linear factor by x^2 in the quadratic factor). So we need another $-2x^2$ to get a total of $-3x^2$. So the x term in the quadratic factor must be $-2x$, so that we get $-2x^2$ by multiplying x in the linear factor by $-2x$ in the quadratic factor.

$$x^3 - 3x^2 - x + 3 = (x-1)(x^2 - 2x - 3)$$

Don't worry if you find this method too difficult at first. It's fine to use the method shown in Example 6, or one of the other methods shown in the interactive resources listed below.

The methods shown in the Flash resources ***Polynomial division by inspection***, ***Factorising a cubic*** and ***Polynomial division – box method*** are similar to the ones shown above. The 'box method' is just another way of setting out the working which some people prefer.

The PowerPoint presentation ***Factorising polynomials*** shows the inspection method and the box method.

Another method is algebraic long division. You can see examples of this method using the Flash resource ***Polynomial long division***.

You can also look at the ***Solving cubics video***.

Example 7
$f(x) = x^3 + px^2 + 11x - 6$ has a factor $x - 2$.
Find the value of p and hence factorise $f(x)$ as far as possible.

Solution

$x - 2$ is a factor of $f(x) \Leftrightarrow f(2) = 0$

$f(2) = 8 + 4p + 22 - 6 = 24 + 4p$

$24 + 4p = 0 \Rightarrow p = -6$

$f(x) = x^3 - 6x^2 + 11x - 6$

$$\begin{aligned} x^3 - 6x^2 + 11x - 6 &= (x-2)(x^2 - 4x + 3) \\ &= (x-2)(x-1)(x-3) \end{aligned}$$