AQA Level 2 Further mathematics Number & algebra

Section 3: Functions and their graphs

Notes and Examples

These notes contain subsections on:

The language of functions

"Start with any positive number. Multiply by 2, and then add 1".

The rule above is an example of a function. A function is a rule which can be applied to any object or number in a particular set, and outputs a new object or number.

When you are working with numbers, a function can usually be expressed algebraically. The function above can be written like this:

$$f(x) = 2x + 1 \qquad x > 0$$

The set of possible inputs for the function is called the domain of the function. In this case, the domain is the set of all positive numbers, so this can be written as $x > 0$.

The set of possible outputs for the function is called the range of the function. In this case, because the domain is $x > 0$, the output must be greater 1, so the range is $f(x) > 1$. If we had chosen a different domain, the range would have been different. For example, we could have chosen "all real numbers" for the domain, in which case the range would also be "all real numbers".

Example 1

The function $f(x)$ is defined as $f(x) = (x - 1)^2 + 3 \qquad x > 0$

(i) Find the value of $f(5)$.

(ii) What is the range of $f(x)$?

(iii) Find an expression for $f(2x + 1)$. Give your answer in its simplest form.

Solution

(i) $f(5) = (5 - 1)^2 + 3 = 4^2 + 3 = 16 + 3 = 19$.

(ii) The smallest possible value of $f(x)$ is 3 (when $x = 1$).
So the range of $f(x)$ is given by $f(x) > 3$.

The expression in the bracket cannot be negative – its smallest possible value is 0

(iii) $\mathrm{f}(2x+1) = (2x+1-1)^2 + 3$
$= (2x)^2 + 3$
$= 4x^2 + 3$

Replace x by $2x + 1$ in the definition of the function

Example 2

The function f(x) is defined as $\mathrm{f}(x) = \dfrac{x}{x+1}$.

(i) Find f(4).

(ii) f(x) has domain all x except $x = a$. What is the value of a?

Solution

(i) $\mathrm{f}(4) = \dfrac{4}{4+1} = \dfrac{4}{5} = 0.8$

(ii) $a = -1$

When $x = -1$, the denominator is zero so the function isn't defined. So $x = -1$ has to be excluded from the domain.

Gradients

You will have probably have met gradients before. To revise finding the gradient of a line from a diagram, use the interactive questions ***The gradient of a line***. Remember that lines which go "downhill" have negative gradients.

To find the gradient of a straight line between two points (x_1, y_1) and (x_2, y_2), use the formula

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example 3

P is the point (-3, 7). Q is the point (5, 1).
Calculate the gradient of PQ

Solution

Choose P as (x_1, y_1) and Q as (x_2, y_2).

or vice versa: it will still give the same answer (***WHY***?)

Gradient of PQ = $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{1-7}{5-(-3)} = \dfrac{-6}{8} = -\dfrac{3}{4}$

Notes:
(1) Draw a sketch and check that your answer is sensible (e.g. has negative gradient).
(2) Check that you get the same result when you choose Q as (x_1, y_1) and P as (x_2, y_2).

For more examples on gradient, look at the Flash resource ***Gradient of a line***. You may also find the Mathcentre video ***The gradient of a straight line segment*** useful.

For further practice in examples like the one above, try the interactive questions ***The gradient of a line between two points*** and ***Collinear points***.

The equation of a straight line

The equation of a straight line is often written in the form $y = mx + c$, where m is the gradient and c is the intercept with the y-axis.

Example 4

Find (i) the gradient and (ii) the y-intercept of the following straight-line equations.

(a) $5y = 7x - 3$ (b) $3x + 8y - 7 = 0$

Solution

(a) Rearrange the equation into the form $y = mx + c$.

$5y = 7x - 3$ becomes $y = \frac{7}{5}x - \frac{3}{5}$

so $m = \frac{7}{5}$ and $c = -\frac{3}{5}$

Note the minus sign

(i) The gradient is $\frac{7}{5}$

(ii) The y-intercept is $-\frac{3}{5}$.

(b) Rearrange the equation into the form $y = mx + c$.

$3x + 8y - 7 = 0$ becomes $8y = -3x + 7$

giving $y = -\frac{3}{8}x + \frac{7}{8}$

so $m = -\frac{3}{8}$ and $c = \frac{7}{8}$

Note the minus sign

(i) The gradient is $-\frac{3}{8}$

(ii) The y-intercept is $\frac{7}{8}$.

Sometimes you may need to sketch the graph of a line. A sketch is a simple diagram showing the line in relation to the origin. It should also show the coordinates of the points where it cuts one or both axes.

You can explore straight line graphs using the Flash resources ***Equation of a line y = mx + c*** and ***Equation of a line ax + by + c = 0***. You may also find the Mathcentre video ***Equations of a straight line*** and ***Linear functions and graphs*** useful.

Example 5

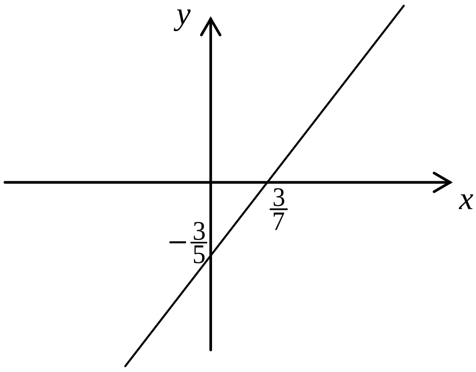
Sketch the lines (a) $5y = 7x - 3$ (b) $3x + 8y - 7 = 0$

Solution

(a) $5y = 7x - 3$

When $x = 0$, $y = -\frac{3}{5}$

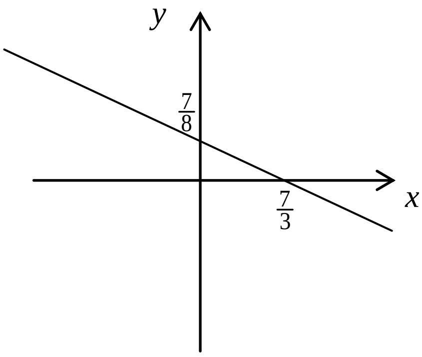
When $y = 0$, $x = \frac{3}{7}$



(b) $3x + 8y - 7 = 0$

When $x = 0$, $y = \frac{7}{8}$

When $y = 0$, $x = \frac{7}{3}$



Sometimes you may need to find the equation of a line given certain information about it. If you are given the gradient and intercept, this is easy: you can simply use the form $y = mx + c$. However, more often you will be given the information in a different form, such as the gradient of the line and the coordinates of one point on the line (as in Example 6) or just the coordinates of two points on the line (as in Example 7).

In such cases you can use the alternative form of the equation of a straight line. For a line with gradient m passing through the point (x_1, y_1), the equation of the line is given by

$$y - y_1 = m(x - x_1).$$

Example 6

Find the equation of the line with gradient 2 and passing through (3, -1).

$m = 2$ and (x_1, y_1) is (3, -1)

Solution

The equation of the line is
$$\begin{aligned} & y - y_1 = m(x - x_1) \\ \Rightarrow\ & y - (-1) = 2(x - 3) \\ \Rightarrow\ & y + 1 = 2x - 6 \\ \Rightarrow\ & y = 2x - 7 \end{aligned}$$

You should check that the point (3, -1) satisfies your line. If it doesn't, you must have made a mistake!

You can see more examples like this using the Flash resource ***Equation of a line y – y1 =m(x – x1)***.

In the next example, you are given the coordinates of two points on the line.

Example 7

P is the point (3, 8). Q is the point (-1, 5).
Find the equation of PQ.

One method is to find the gradient and then use this value and one of the points in $y - y_1 = m(x - x_1)$

Solution

Choose P as (x_1, y_1) and Q as (x_2, y_2).

Gradient of PQ = $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5 - 8}{-1 - 3} = \dfrac{-3}{-4} = \dfrac{3}{4}$

Now use
$$\begin{aligned} & y - y_1 = m(x - x_1) \\ \Rightarrow\ & y - 8 = \tfrac{3}{4}(x - 3) \\ \Rightarrow\ & 4(y - 8) = 3(x - 3) \\ \Rightarrow\ & 4y - 32 = 3x - 9 \\ \Rightarrow\ & 4y = 3x + 23 \end{aligned}$$

You should check that P and Q satisfy your line.

An alternative approach to the above examples is to put the formula for m into the straight line equation to obtain
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

and then make the substitutions. This is equivalent to the first method, but does not involve calculating m separately first.

For further practice in examples like the one above, try the interactive questions ***The equation of a line between two points***.

The intersection of two lines

The point of intersection of two lines is found by solving the equations of the lines simultaneously. This can be done in a variety of ways. When both equations are given in the form $y = \ldots$ then equating the right hand sides is a good approach (see below). If both equations are not in this form, you can re-arrange them into this form first, then apply the same method. Alternatively, you can use the elimination method if the equations are in an appropriate form.

Example 8
Find the point of intersection of the lines $y = 3x - 2$ and $y = 5x - 8$.

Solution

$3x - 2 = 5x - 8$

$\Rightarrow -2 = 2x - 8$

$\Rightarrow 6 = 2x$

$\Rightarrow x = 3$

Substitute into one of the equations to find y

Substituting $x = 3$ into $y = 3x - 2$ gives $y = 3 \times 3 - 2 = 7$

The point of intersection is (3, 7)

Check that (3, 7) satisfies the second equation.

You can see more examples like this using the Flash resource ***Intersection of two lines***.

Sketching the graphs of functions

You can sketch the graph of a linear or quadratic function by thinking about where the graph cuts the coordinate axes.

- A linear function is of the form $f(x) = ax + b$. The graph of $y = f(x)$ is a straight line. You can find where it cuts the y-axis by substituting $x = 0$, and you can find where it cuts the x-axis by substituting $y = 0$, as in Example 5.

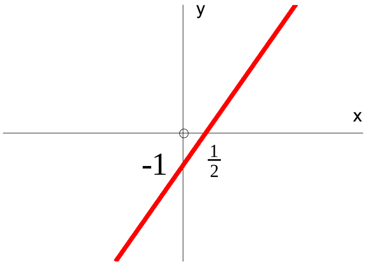
- A quadratic function is of the form $f(x) = ax^2 + bx + c$. The graph of $y = f(x)$ is a curve called a parabola. You can find where the graph cuts the y-axis by substituting $x = 0$. You can find where it cuts the x-axis by solving the equation $ax^2 + bx + c = 0$. If the equation has no solutions, then the graph does not cut the x-axis.

Example 9
Sketch the graph of $y = f(x)$ for each of the following functions:
(i) $f(x) = 2x - 1$
(ii) $f(x) = x^2 - x - 2$

(iii) $f(x) = x^2 + 1$

Solution

(i) This is a straight line graph.
$f(0) = -1$ so the graph crosses the y-axis at $(0, -1)$
When $y = 0$, $2x - 1 = 0$ so $x = \frac{1}{2}$, so the graph crosses the x-axis at $\left(\frac{1}{2}, 0\right)$.

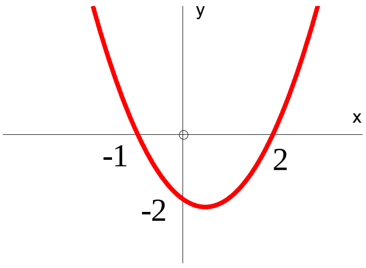


(ii) $f(0) = -2$, so the graph crosses the y-axis at $(0, -2)$
When $y = 0$, $x^2 - x - 2 = 0$

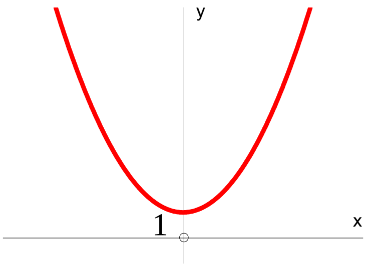
$$(x-2)(x+1) = 0$$

$$x = 2 \text{ or } -1$$

so the graph crosses the x-axis at $(2, 0)$ and $(-1, 0)$



(iii) $f(0) = 1$, so the graph crosses the y-axis at $(0, 1)$
When $y = 0$, $x^2 + 1 = 0$ has no solutions, so the graph does not cut the x-axis.



Functions are sometimes defined 'piecewise' – this means that the domain is split into separate parts and the function is defined differently for each part. Here is an example.

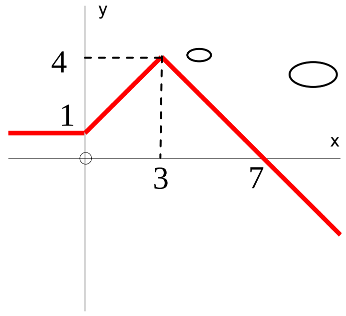
Example 10

A function f(x) is defined as

$$
\begin{aligned}
\mathrm{f}(x) &= 1 & x &< 0 \\
&= x+1 & 0 \le x &< 3 \\
&= 7-x & x &\ge 3
\end{aligned}
$$

Sketch the graph of y = f(x)

Solution



At x = 3, both x + 1 and 7 – x have the value 4