Section 1: Basic number and algebra

Notes and examples

These notes contain subsections on

Number

You need to be confident in working with numbers, including fractions, decimals, ratio and percentages. The notes and examples below remind you of some of the skills you need.

Fractions

- When adding or subtracting fractions, you need to use a common denominator. For mixed fractions you can deal with the whole numbers separately – but make sure you tidy up at the end if the fraction part is top-heavy.
- To multiply fractions, mixed numbers must be changed into top-heavy fractions before multiplying the numerators and the denominators. It's easier to do any 'cancelling' before doing the multiplication.
- When dividing, you must also change mixed numbers into top-heavy fractions. Dividing by a fraction is the same as multiplying by its reciprocal.

Example 1
Work out:

(i) $2\frac{3}{4}+1\frac{2}{3}$

(ii) $5\frac{1}{6}-2\frac{5}{8}$

(iii) $2\frac{1}{4}\times 1\frac{2}{3}$

(iv) $1\frac{1}{3}\div 1\frac{3}{5}$

Solution

(i) $2\frac{3}{4}+1\frac{2}{3}=2+\frac{9}{12}+1+\frac{8}{12}$

$$=3\frac{17}{12}$$

$$=4\frac{5}{12}$$

(ii) $5\frac{1}{6}-2\frac{5}{8}=5+\frac{4}{24}-2-\frac{15}{24}$

$$=3-\frac{11}{24}$$

$$=2+\frac{24}{24}-\frac{11}{24}$$

$$=2\frac{13}{24}$$

(iii) $2\frac{1}{4}\times1\frac{2}{3}=\frac{9}{4}\times\frac{5}{3}$

$$=\frac{\cancel{9}^{3}}{4}\times\frac{5}{\cancel{3}^{1}}$$

$$=\frac{15}{4}$$

$$=3\frac{3}{4}$$

(iv) $1\frac{1}{3}\div1\frac{3}{5}=\frac{4}{3}\div\frac{8}{5}$

$$=\frac{\cancel{4}^{1}}{3}\times\frac{5}{\cancel{8}^{2}}$$

$$=\frac{5}{6}$$

Percentages

- To work with percentages, you will usually need to write them as decimals.
- For example, to find 58% of something, you need to multiply by 0.58
- To increase something by, say, 8%, multiply by 1.08 (the 1 gives you the original amount, and the 0.08 gives the extra 8%)
- To decrease something by, say, 24%, multiply by 0.76 (the new amount is 76% of the original amount.

Example 2

(i) Find 6% of 240.

(ii) Amy scores 68 marks out of 80 in a test. What is this as a percentage?

(iii) A restaurant bill is £32.40. I have a voucher which gives a 15% discount. How much do I pay?

(iv) Ahmed is paid £7.60 per hour. His pay is increased to £8.20 per hour. What is the percentage increase in his pay (to the nearest whole number)?

Solution

(i) 6% as a decimal is 0.06.

$240 \times 0.06 = 14.4$

(ii) To change a fraction into a percentage, multiply by 100.

$\frac{68}{80} \times 100 = 85$

(iii) 15% is taken off the bill, so the final amount is 85% of the original bill.

$32.4 \times 0.85 = 27.54$

The bill is £27.54

(iv) Ahmed's pay has been multiplied by $\frac{8.20}{7.60} = 1.078...$

As a percentage this is 108% to the nearest whole number.

Ahmed's pay is 108% of the previous amount, so it has been increased by 8%.

Ratio

- Ratios are similar to fractions, in that you can 'cancel' them down to simplify them.

Example 3

Given that $x : y = 2 : 3$ and $y : z = 2 : 5$, find:

(i) $x : z$

(ii) $2y : 3z$

(iii) $x + y : z$

Solution

(i) $x : y = 4 : 6$

$y : z = 6 : 15$

so $x : z = 4 : 15$

(ii) $2y : 3z = 4 : 9$

(iii) $x + y : y = 5 : 3 = 10 : 6$

$y : z = 6 : 15$

$x + y : z = 10 : 15 = 2 : 3$

Manipulating algebraic expressions

Throughout your mathematics you will need to be able to manipulate algebraic expressions confidently. The examples below remind you of the important techniques of collecting like terms, removing brackets, factorising, multiplying, and adding, subtracting and simplifying algebraic fractions.

Collecting like terms

Example 4
Simplify the expression
$3a + 2b - a + 3b - 2ab + 2a$

Solution
There are three different types of "like term" in this expression. There are terms in a, terms in b, and a term in ab. Notice that the term in ab cannot be combined with either the terms in a or the terms in b, but remains as a term on its own.

$$\begin{aligned} 3a + 2b - a + 3b - 2ab + 2a &= 3a - a + 2a + 2b + 3b - 2ab \\ &= 4a + 5b - 2ab \end{aligned}$$

In the example the expression has been rewritten with each set of like terms grouped together, before simplifying by adding / subtracting the like terms. You may well not need to write down this intermediate stage.

For practice in examples like this one, try the interactive questions ***Collecting terms***.

Expanding brackets

When multiplying out brackets, each term in the bracket must be multiplied by the number or expression outside the bracket.

Example 5
Simplify the expressions
(i) $3(p - 2q) + 2(3p + q)$
(ii) $2x(x + 3y) - y(2x - 5y)$

Solution
(i) $$\begin{aligned} 3(p - 2q) + 2(3p + q) &= 3p - 6q + 6p + 2q \\ &= 9p - 4q \end{aligned}$$
(ii) $$\begin{aligned} 2x(x + 3y) - y(2x - 5y) &= 2x^2 + 6xy - 2xy + 5y^2 \\ &= 2x^2 + 4xy + 5y^2 \end{aligned}$$

Multiplying out two brackets of the form $(ax + b)(cx + d)$ gives a quadratic function. Make sure that you are confident in this.

Example 6
Multiply out $(x+2)(3x-4)$

You need to multiply each term in the first bracket by each term in the second. Use FOIL – First, Outer, Inner, Last.

Solution

$$(x+2)(3x-4) = 3x^2 - 4x + 6x - 8$$
$$= 3x^2 + 2x - 8$$

You can see a step-by-step version of this example in the PowerPoint presentation ***Multiplying out brackets***.

You can also look at the ***Expanding brackets video***, which shows multiplying out brackets, starting from an investigation that you might have seen before. It then looks at multiplying out first just one bracket, then two brackets. It then goes on to deal with some harder examples, which are beyond the requirements of this section but will be covered in the chapter on Polynomials, so are well worth looking at. The whole video lasts 40 minutes, so you may wish to fast-forward over parts of it if your time is limited.

You can test yourself on multiplying out brackets using the Flash resource ***Expanding brackets***.

You might also need to multiply out more complicated examples, such as multiplying a quadratic expression by a linear expression. The important point to remember is that each term in the first bracket must be multiplied by each term in the second bracket. So (before simplifying) you can check that you have the right number of terms by multiplying the number of terms in one bracket by the number of terms in the other bracket.

Example 7
Multiply out $(2x - 3)(x^3 - 2x^2 + 3x + 4)$

There are 2 terms in the first bracket and 4 in the second, so there will be 8 terms in the expansion

Solution

$$(2x-3)(x^3-2x^2+3x+4) = 2x(x^3-2x^2+3x+4) - 3(x^3-2x^2+3x+4)$$
$$= 2x^4 - 4x^3 + 6x^2 + 8x - 3x^3 + 6x^2 - 9x - 12$$
$$= 2x^4 - 7x^3 + 12x^2 - x - 12$$

You can see more examples using the Flash resource ***Multiplying polynomials***.

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You also need to be able to multiply out expressions involving more than one bracket. This is shown in the next example.

Example 8
Multiply out $(x-2)(2x+3)(3x-1)$

Solution
It is often easiest to multiply out one pair of brackets, and then multiply the result by the third bracket.

$$\begin{aligned}(x-2)(2x+3) &= 2x^2+3x-4x-6\\ &= 2x^2-x-6\end{aligned}$$

Simplify here or the next step will be more complicated!

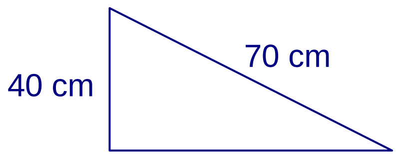
$$\begin{aligned}(2x^2-x-6)(3x-1) &= 2x^2(3x-1)-x(3x-1)-6(3x-1)\\ &= 6x^3-2x^2-3x^2+x-18x+6\\ &= 6x^3-5x^2-17x+6\end{aligned}$$

An **irrational number** is a number which cannot be expressed as one whole number divided by another whole number.

Rational and irrational numbers

The square root of any number which is not itself a perfect square is an **irrational number**. So $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers, but $\sqrt{4}$ is not as it is equal to 2, which is a rational number. A number which is partly rational and partly square root (or cube root etc.) is called a **surd**. (There are of course other irrational numbers which do not involve a root, such as π.

In this section you will learn to manipulate and simplify expressions involving surds. This is an important skill in many areas of mathematics. For example, suppose you have a triangular paving slab like this:



You can use Pythagoras' theorem to work out that the length of the third side is $\sqrt{3300}$. You could use a calculator to work this out and give the answer to two or three decimal places, but this would no longer be exact. Suppose you wanted to find the area and perimeter of the slab, or the total area of 100 slabs, or find out how many slabs you could make from a certain volume of concrete? It is much better to use the exact answer in these calculations, and then the results will also be exact.

Writing a square root in terms of a simpler square root

Square roots like the one in the example above look quite daunting and can be difficult to work with. However, many square roots can be written in terms of a simpler square root like $\sqrt{2}$ or $\sqrt{3}$ (and the same applies to cube roots and so on). Example 9 shows how to do this.

Example 9
Write these numbers in terms of the simplest possible square roots.
(i) $\sqrt{12}$ (ii) $\sqrt{72}$ (iii) $\sqrt{150}$

Solution
(i) $12 = 4 \times 3$
$$\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$$

Look for any factors which are perfect squares

(ii) $72 = 36 \times 2$
$$\sqrt{72} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}$$

(iii) $150 = 25 \times 6$
$$\sqrt{150} = \sqrt{25} \times \sqrt{6} = 5\sqrt{6}$$

Adding and subtracting surds

Adding and subtracting surds is rather like adding or subtracting algebraic expressions, in that you have to collect "like terms". You should collect together any rational numbers, and collect together any terms involving roots of the same number. You cannot collect together terms involving roots of different numbers, such as $\sqrt{2}$ and $\sqrt{3}$.

Example 10
Simplify
(i) $(2+\sqrt{2})+(3-2\sqrt{2})$
(ii) $(4-\sqrt{3})-(1-2\sqrt{2}+3\sqrt{3})$
(iii) $\sqrt{32}-\sqrt{18}$

Solution
(i)
$$\begin{aligned}(2+\sqrt{2})+(3-2\sqrt{2}) &= 2+3+\sqrt{2}-2\sqrt{2}\\ &= 5-\sqrt{2}\end{aligned}$$

(ii)
$$\begin{aligned}(4-\sqrt{3})-(1-2\sqrt{2}+3\sqrt{3}) &= 4-\sqrt{3}-1+2\sqrt{2}-3\sqrt{3}\\ &= 4-1-\sqrt{3}-3\sqrt{3}+2\sqrt{2}\\ &= 3-4\sqrt{3}+2\sqrt{2}\end{aligned}$$

(iii) $\sqrt{32}-\sqrt{18}$

$$\sqrt{32}=\sqrt{16}\times\sqrt{2}=4\sqrt{2}$$
$$\sqrt{18}=\sqrt{9}\times\sqrt{2}=3\sqrt{2}$$
$$\sqrt{32}-\sqrt{18}=4\sqrt{2}-3\sqrt{2}$$
$$=\sqrt{2}$$

At first sight this looks like it cannot be simplified. However, both surds can be written in terms of simpler surds (as in Example 9)

For further practice in examples like the one above, use the interactive resource ***Adding and subtracting surds***.

Multiplying surds

Multiplying two or more square roots is quite simple – you just multiply the numbers. You may then be able to write the result as a simpler surd.

Example 11
Simplify
(i) $\sqrt{2}\times\sqrt{6}$
(ii) $2\sqrt{15}\times\sqrt{6}\times3\sqrt{10}$

Solution
(i) $\sqrt{2}\times\sqrt{6}=\sqrt{12}$
$$=\sqrt{4}\times\sqrt{3}$$
$$=2\sqrt{3}$$

The approach used in (i) works well for small numbers, but for (ii) you would get the square root of a large number to simplify. An easier way for this example is to split each surd into simpler ones and then look for any pairs.

(ii) $2\sqrt{15}\times\sqrt{6}\times3\sqrt{10}=(2\times\sqrt{3}\times\sqrt{5})\times(\sqrt{2}\times\sqrt{3})\times(3\times\sqrt{5}\times\sqrt{2})$
$$=2\times3\times\sqrt{3}\times\sqrt{3}\times\sqrt{5}\times\sqrt{5}\times\sqrt{2}\times\sqrt{2}$$
$$=6\times3\times5\times2$$
$$=180$$

The next example deals with multiplying expressions involving a mixture of rational numbers and roots. You have to use brackets for this, and it is very similar to multiplying out two brackets in algebra – each term in the first bracket needs to be multiplied by each term in the second bracket. You can use FOIL (First, Outer, Inner, Last) if it helps you.

Example 12
Multiply out and simplify
(i) $(2+\sqrt{3})(1-2\sqrt{3})$
(ii) $(3-\sqrt{2})^2$

(iii) $(\sqrt{5}-2)(\sqrt{5}+2)$

Solution

(i) $(2+\sqrt{3})(1-2\sqrt{3}) = 2-4\sqrt{3}+\sqrt{3}-2\sqrt{3}\times\sqrt{3}$

$$= 2-3\sqrt{3}-6$$

$$= -4-3\sqrt{3}$$

(ii) $(3-\sqrt{2})^2 = (3-\sqrt{2})(3-\sqrt{2})$

$$= 9-3\sqrt{2}-3\sqrt{2}+\sqrt{2}\times\sqrt{2}$$

$$= 9-6\sqrt{2}+2$$

$$= 11-6\sqrt{2}$$

(iii) $(\sqrt{5}-2)(\sqrt{5}+2) = \sqrt{5}\times\sqrt{5}+2\sqrt{5}-2\sqrt{5}-2\times 2$

$$= 5-4$$

$$= 1$$

For further practice in examples like the one above, try the interactive resources ***Multiplying surds*** and ***Squaring and cubing surds***.

You can see some mixed examples on surds using the Flash resource ***Working with surds***.

Part (iii) of Example 12 illustrates a very important and useful idea. Multiplying out any expression of the form $(a+b)(a-b)$ gives the result a^2-b^2. The "outer" and "inner" products, $-ab$ and ab, cancel each other out. When either or both of a and b are surds, the result a^2-b^2 is a rational number.

Rationalising the denominator

Surds in the denominator of a fraction can be a real nuisance! However, you can get rid of them from the denominator by a process called *rationalising the denominator*, which uses the idea above. If the denominator of a fraction is $a+b$, where either or both of a and b are surds, then you can multiply both top and bottom of the fraction by $a-b$. The denominator is then $(a+b)(a-b)$, which works out to be rational. *Since you multiply the top and bottom of the fraction by the same amount, its value is unchanged*. The numerator still involves surds, but this is not quite so difficult to work with.

This technique is shown in Example 13.

Example 13

Simplify the following by rationalising the denominator.

(i) $\dfrac{1}{\sqrt{3}}$ (ii) $\dfrac{1}{\sqrt{2}-1}$ (iii) $\dfrac{2+\sqrt{3}}{1-\sqrt{3}}$

Solution

(i) $\dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{\sqrt{3}}{3}$

Multiply top and bottom by $\sqrt{3}$

(ii) $\dfrac{1}{\sqrt{2}-1} = \dfrac{1}{\sqrt{2}-1} \times \dfrac{\sqrt{2}+1}{\sqrt{2}+1}$

$= \dfrac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)}$

$= \dfrac{\sqrt{2}+1}{2-1}$

$= \sqrt{2}+1$

Multiply top and bottom by $\sqrt{2}+1$

(iii) $\dfrac{2+\sqrt{3}}{1-\sqrt{3}} = \dfrac{2+\sqrt{3}}{1-\sqrt{3}} \times \dfrac{1+\sqrt{3}}{1+\sqrt{3}}$

$= \dfrac{(2+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$

$= \dfrac{2+2\sqrt{3}+\sqrt{3}+3}{1-3}$

$= -\tfrac{1}{2}(5+3\sqrt{3})$

Multiply top and bottom by $1+\sqrt{3}$

You can see some similar examples using the Flash resource ***Rationalising the denominator***. You may also find the Mathcentre video ***Surds*** useful.

For further practice in examples like the one above, try the interactive questions ***Rationalising the denominator***.

To test your understanding of this section, try the ***Surds Puzzle***, either on your own or with a group of friends.